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## *Calculus of Direction and Position.*

BY E. W. HYDE, *Cincinnati.*

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It appears to me to be obvious that the calculus of directed quantities in some form ought to and will come more and more into use in all the operations of Geometry and Mechanics, owing to its peculiar fitness for expressing the relations and conditions of the quantities discussed in these branches of science. It is therefore important that the *best* and most *natural* system should be employed, and taught to the students who are to form the next generation of mathematicians and scientific men.

I propose in the present paper to give briefly the fundamental idea and principles of Hamilton's system, or "Quaternions," and of Grassmann's, called by him "Die Ausdehnungslehre," showing their points of difference, and what seem to me to be strong reasons why the system of Grassmann is far preferable.

The title of this paper indicates at once a most important difference between the two systems, for Quaternions is a calculus of magnitude and direction *only*, while the "Ausdehnungslehre" is a calculus of magnitude, direction and *position*. This is precisely what is required for Mechanics, for a force is fully determined only when these three things are known concerning it. It also greatly facilitates many operations of Geometry, as will appear in the sequel.

Hamilton's ruling idea in forming his system was *rotation*, and his versors are really  $\sqrt{-1}$ , endowed with a directed quality so as to turn the vector operated on about some particular *axis*. The numerical quantity  $\sqrt{-1}$  may be called an *undirected* versor.

The *addition* and *subtraction* of directed quantities was understood before the invention of Quaternions, the peculiarity of which depends on its method of *multiplication*.

The basis of the whole theory may be given in a few words as follows :

Taking  $I, J, K$  as three mutually perpendicular unit vectors, and  $i, j, k$  as operators or versors that turn  $J$  into  $K$ ,  $K$  into  $I$  and  $I$  into  $J$  respectively, it is shown that

$$\left\{ \begin{array}{l} ij = k \\ iJ = K \end{array} \right\}, \quad \left\{ \begin{array}{l} ji = -k \\ jI = -K \end{array} \right\} \text{ etc.}$$

(*not*, however, that  $IJ = K$ ), and then it is *assumed* that it is permissible to take  $i, j, k$  as *identical* with  $I, J, K$ , which is certainly true, the only question being whether it is on the whole *best* to do so.

Tait says (Art. 72, Tait's Quat.): "Now the meanings we have assigned to  $i, j, k$ , are quite independent of, and not inconsistent with, those assigned to  $I, J, K$ . And it is superfluous to use two sets of characters where one will suffice. Hence it appears that  $i, j, k$  may be substituted for  $I, J, K$ ; in other words, *a unit-vector when employed as a factor may be considered as a quadrantal versor whose plane is perpendicular to the vector*. This is one of the main elements of the singular simplicity of the quaternion calculus."

This last statement I entirely disagree with.

Again, in Art. 64, Tait says: "We shall content ourselves at present with an assumption, which will be shown to lead to consistent results; but at the end of the chapter we shall show that no other assumption is possible, following for this purpose a very curious quasi-metaphysical speculation of Hamilton."

The statement "that no other assumption is possible" I deny, and I propose to show that, though Hamilton's reasoning is correct, the deductions therefrom are unwarranted.

The "speculation" referred to above, as given in abridged form by Tait, is as follows (see Art. 93, Tait's Quat.): "Suppose that no direction in space is preëminent, and that the product of two vectors is something that has quantity, so as to vary in amount if the factors are changed, and to have its sign changed if that of one of them is reversed; if the vectors be parallel, their product cannot be, in whole or in part, a vector *inclined* to them, for there is nothing to determine the direction in which it must lie. It cannot be a vector *parallel* to them; for by changing the sign of both factors the product is unchanged, whereas, as the whole system has been reversed, the product vector ought to have been reversed. Hence it must be a number. Again, the product of two perpendicular vectors cannot be wholly or partly a number, because on inverting one of them the sign of that number ought to change; but inverting one of them is simply equivalent to a rotation through two right angles about the other, and (from the symmetry of space) ought to leave the number unchanged. Hence the product of two perpendicular vectors must be a vector, and a simple extension of the same reasoning shows that it must be perpendicular to each of the factors."

Now the reasoning as to the product of  $\parallel$  vectors, *i. e.* that it is scalar, is perfectly correct, but it does not follow that this product must be — the product of their tensors, as in quaternions; it may, for instance, be zero, as will be shown hereafter.

The reasoning also with regard to the product of two  $\perp$  vectors is correct, if we write *vector quantity* or *directed quantity* instead of vector in the last sentence. But a plane area is a quantity having magnitude and direction as well as a portion of a right line, so that there is nothing in the reasoning which precludes  $ij$  from meaning a square unit of plane area  $\parallel$  to  $i$  and  $j$ , which certainly appears a more natural signification than that  $ij$  should be *equal* to  $k$ , a unit vector  $\perp$  to  $i$  and  $j$ . From this assumption it follows as above, that  $ij = k$  and also that  $i/j = -ij = -k$ , *i. e.* the *ratio* of two quantities is the same thing as their *product* except as to sign. To be sure we may say that these are *units*, and we have the analogy that  $1/1 = 1 \times 1$ ; but they, *i. e.* vectors, are *geometric* and *directed* units, and such a relation appears to me to upset all one's preconceived ideas of geometric quantities without any corresponding advantage. If, in the eq.  $1/1 = 1 \times 1$ , 1 be taken as a unit of *length*, then the members of the equation have evidently not the same meaning,  $1/1$  being merely a numerical quantity while  $1 \times 1$  is a unit of *area*, it being a fundamental geometric conception that the product of a length by a length is an *area*, that of a length by an area a volume, while the ratio of two quantities of the same order as that of a length to a length is a mere number of the order zero. In quaternions however we have the remarkable result that the product of a length by a length is not merely represented by, but actually *equal* to a length  $\perp$  to the plane of the two.

Of course this arises from the double function of the vector as used in quaternions, it being not only something endowed with magnitude and direction, but also possessing the properties of the versor  $\sqrt{-1}$ .

The combination of these different functions in the vector renders the product of two vectors which are neither  $\parallel$  nor  $\perp$  to each other necessarily a *complex* quantity, having a scalar and a vector part corresponding to the real and imaginary parts of the ordinary complex  $a + b\sqrt{-1}$ , thus making a thing which should be simple just the opposite.

It seems to me that quaternions proper, *i. e.* these complex quantities, are practically of little use. In nearly all the applications to geometry and mech-

anics, scalars or vectors are used *separately*. For the special uses to which the complex  $a + b\sqrt{-1}$  is put, the directed quality is not needed.

Another point about this system appears objectionable to my mind, viz. that we must *necessarily* work in space of three dimensions. Even when nominally treating plane geometry, the product of any two vectors in the plane is a vector  $\perp$  to it, and we are therefore really treating space.

We will now consider Grassmann's system, giving first the way in which he was led to his method of multiplying directed quantities, as stated by himself in the preface to his first book, published in 1844, and then giving a brief account of the whole system.

In the above-mentioned preface he first states how he was led to the addition and subtraction of vectors (*strecken*), and afterwards to their multiplication. The addition and subtraction being precisely as in quaternions, we will not here consider them.

Grassmann arrived at his conception of the product of two *directed* lines from a consideration of the geometric meaning of the product of two *undirected* lines, viz. the rectangle having these two lines for two of its conterminous sides. It would follow at once from analogy that the product of two directed lines at right angles should be the same rectangle as before, endowed with the additional property of *direction*, i. e. it would be a plane area  $\parallel$  to the two given vectors and numerically equal to the product of their lengths. A further extension of the analogy would indicate that the product of *any* two directed lines should be the area of the parallelogram of which these lines are two adjacent sides, or an equal parallelogram *parallel* to this; and similarly the product of *three* vectors should be the volume of the parallelopiped of which they form three conterminous edges. These conceptions were found to work consistently, and it appeared that this species of multiplication agreed with ordinary multiplication in being subject to the associative and distributive laws, but different from it in that it did *not* obey the commutative law.

Thus, if  $a$ ,  $b$ ,  $c$  are three non-co-planar vectors, we have  $abc = a.bc = ab.c$ ,  $ab + ac = a(b + c)$ , but *not*  $ab = ba$ . Instead of the last we have

$$ab = -ba \text{ or } ab + ba = 0.$$

It follows at once from this non-commutative law, as well as from the meaning of the product of two vectors given above, that  $aa = 0$ .

Similarly  $aab = 0$ ,  $aba = 0$ , etc.

These are the laws of what Grassmann calls "outer multiplication," and I think their simplicity will be acknowledged as compared with the multiplication of vectors in quaternions.

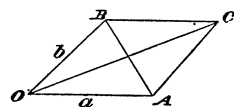
The conception formed by Grassmann of the product  $ab$  was that the vector  $a$  generates the area  $ab$  by moving parallel to itself along the vector  $b$  as a directrix from its initial to its final point. This idea of generation by a moving point, line or surface led to his use of the term "ausdehnung" or "extension" as descriptive of his system.

We will now give a brief sketch of Grassmann's method of treating points and vectors.

A point is some position in space with a value attached to it which we may call its *weight* (using the term somewhat as in the theory of least squares), and which may be positive or negative. We may use a letter, as  $A$ , to represent a unit point, *i. e.* a point of unit weight, and then a point whose weight is  $m$  will be  $mA$ .

Two unit points can differ only in *position*, *i. e.* by a certain length of straight line in a certain direction, or, in other words, by a *vector*. Hence if  $A$  and  $B$  are two unit points, and  $a$  a vector drawn from  $A$  to  $B$ , we have  $B - A = a$ . Also at once if  $B - O = b$  and  $A - O = a$

$B - A = B - O - A + O = B - O - (A - O) = b - a$   
and  $a + b = A - O + B - O = A - O + C - A = C - O$ .



Since points and vectors can thus be added and subtracted they must be quantities of the same kind, and in fact a vector is a point of weight zero situated at infinity, as will presently appear.

The sum of two points as  $mA$  and  $nB$  must be something of the same kind, and therefore a *point*, and its weight should evidently be equal to the sum of the weights of the two points, *i. e.*  $m + n$ : hence write

$$mA + nB = (m + n)S,$$

$S$  being a unit point. To find the position of  $S$  subtract  $(m + n)O$  from both sides of the above equation,  $O$  being any unit point whatever.

$$m(A - O) + n(B - O) = (m + n)(S - O), \text{ or}$$

$$S - O = \frac{m(A - O) + n(B - O)}{m + n};$$

by which  $S$  may be constructed. This equation shows that the sum of two points is their *mean point*. Or it may be put thus: If  $m$  and  $n$  are regarded as parallel forces acting at  $A$  and  $B$ , then  $S$  is the centre of these parallel forces.

The same holds for any number of points.

Next suppose  $m + n = 0$ , or  $m = -n$ ; then we have

$$mA + nB = n(B - A) = 0. S = 0. (S - O).$$

To satisfy these equations we must either have  $m = n = 0$ , or  $A = B$ , *i. e.*  $A$  the same point as  $B$ , or else  $S - O = \infty$ , *i. e.*  $S$  situated at  $\infty$ ; thus it appears as stated above that a vector is a point at  $\infty$ .

Passing now to multiplication, we will start with two or three definitions, and a statement of the geometric meaning of outer multiplication more general than that previously given, and from a different point of view.

Two posited quantities differ in position when they have no point in common, and not otherwise.

According to this definition, parallel right lines or a plane and parallel line do not differ in position, as in each case there is a common point at  $\infty$ .

A *point* is said to be of the *first order*, the product of *two* points of the *second order*, etc. (*order* corresponds to Grassmann's *stufe*). The locus of all points which can be expressed in terms of  $n$  given points is called a *region* of the  $n^{\text{th}}$  order. Thus a plane is a region of the third order; space of three dimensions a region of the fourth order.

*The outer product of two posited quantities which differ in position is some multiple of the connecting figure.*

If two posited quantities do *not* differ in position, the connecting figure will be zero, and therefore the product zero. This is true whenever the sum of the orders of the two factors is not greater than the order of the region under consideration. Or, in other words, whenever the two factors are such that it is *possible* for them to differ in position in the region under consideration.

For instance, if we are considering space, or the region of the fourth order, two right lines *can* differ in position, while if we are dealing with a *plane* region they *cannot*.

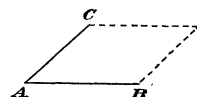
*The outer product of two posited quantities which cannot differ in position in the region under consideration is the common figure multiplied by a scalar quantity.*

Applying these principles we have for the product of two points that  $\overset{A}{\times} \text{---} \overset{B}{\times} \text{---}$   $AB$  is the line from  $A$  to  $B$ ; that is, it is a portion of the right line fixed by the two points  $A$  and  $B$ , whose length is equal to the distance from  $A$  to  $B$ . It may be situated anywhere on the line. These are exactly the conditions that completely determine a *force*.  $AB$  differs from  $B - A$ , in that the latter may be any *parallel* line of the same length.

If  $B - A = a$  we have  $Aa = A(B - A) = AB$ , so that multiplying a vector by a point fixes its position by making it pass through the point.

Grassmann gives the name "Linientheil" to the product of two points, or a point and a vector, but it appears to me that an appropriate and expressive name in English would be *point-vector*.

The product  $ABC$  is *twice* the connecting triangle, i. e. the parallelogram of which  $A, B, C$  are three corners.



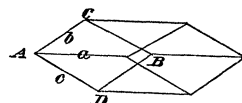
That the product should be twice the connecting figure appears from the previous way of looking at outer multiplication: for  $AB$  is generated by a point in moving from  $A$  to  $B$ , and  $ABC$  is similarly generated by a right line of length  $AB$  moving parallel to itself from  $A$  to  $C$ .

We have, if  $B - A = a$  and  $C - A = b$ ,

$$ABC = AB(C - A) = ABb = A(B - A)b = Aab.$$

I have in my lectures on this subject called the product of two vectors, as  $ab$ , a *plane-vector*, and the product  $ABC = ABb = Aab$  a *point-plane-vector*; the difference being that the latter is fixed in position in so far as that it lies in a fixed plane, while the former may lie in any parallel plane.

The product  $ABCD$  is six times the connecting tetrahedron, that is, the parallelepiped of which  $A, B, C, D$  are four vertices.



As before we may write  $ABCD = ABCc = ABbc = Aabc$ , but in this case each of these is equal to  $abc$ , since there is only *one* space of three dimensions.

As these products can have no direct or posited quality they are *scalars*.

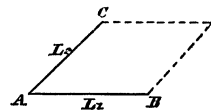
In the region of the fourth order, or space, we have

$$ABCD = ABC.D = A.BCD = AB.CD = L_1L_2,$$

if we put  $L_1 = AB$  and  $L_2 = CD$ , so that the product of two point-vectors is six times the tetrahedron of which they are opposite edges.

In dealing with a *plane region* (as in plane geometry) all the rest of space except this plane may be regarded for the time being as non-existent, so that plane-vectors and point-plane-vectors lose their property of direction and can differ only in magnitude and sign; hence they are *scalar* quantities, a fact which considerably facilitates the application of this method to plane geometry. The difference between a plane-vector and a point-plane-vector also disappears, since but *one* plane is under consideration, so that we have  $Aab = ab = TaTb \sin \frac{b}{a}$ .

We will now look at the product of some quantities which *cannot* differ in position. In a *plane-region*, if  $L_1$  and  $L_2$  are two point-vectors, they cannot differ in position. Let  $A$  be the common point of  $L_1$  and  $L_2$ , and let  $B$  and  $C$  be so taken that  $AB = L_1$  and  $AC = L_2$ ; then  $L_1 L_2 = AB.AC = ABC.A$ : that is, the product is the common point multiplied by the scalar  $ABC$ , which accords with the definition previously given.



In space, let  $L = AB$  and  $P = ACD$ ; then  $LP = AB.ACD = ABCD.A =$  the common point multiplied by the scalar  $ABCD$ .

Similarly, if  $P_1 = ABC$  and  $P_2 = ABD$ ,  $P_1 P_2 = ABC.ABD = ABCD.AB =$  the common point-vector multiplied by the scalar  $ABCD$ .

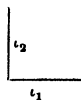
Also  $ab.ac = abc.a$ , *i. e.* the product of two plane-vectors is a vector  $\parallel$  to each of them multiplied by a volume.

In most cases in the use of these products in geometry the scalar coefficients may be disregarded, and we may consider  $AB$  as the line passing through  $A$  and  $B$  of indefinite length,  $ABC$  as the plane through  $A$ ,  $B$  and  $C$ ,  $L_1 L_2$  (in a plane region) as the common point only,  $P_1 P_2$  as the common line of these two planes, etc.

We have thus a simple and complete system of geometric multiplication, in which every product has a clear and definite meaning to be seen at a glance, contrasting thus strongly with many quaternion expressions. What geometric meaning, for instance, can be easily assigned to the expression  $V.a\beta\gamma$ ? Of course there *is* such a meaning which can be gotten at with labor enough, but it is anything but an *evident* meaning. Or let  $a, b, c, d$  be four vectors and compare the two equivalent expressions  $ab.cd$  and  $V.VabVcd$ ; the meaning of the first is seen at a glance, while that of the second requires a considerable mental operation to determine it, to say nothing of the additional labor of writing it.

To complete this brief review of Grassmann's system we have only to consider what he denominates "inner multiplication."

In a plane region any vector may be expressed in terms of two given vectors; if these be unit vectors at right angles we have a *unit normal reference system*. In such a system let the two reference vectors be  $\iota_1$  and  $\iota_2$ , then  $\iota_2$  is the *complement* of  $\iota_1$ , written  $/\iota_1$ , and  $-\iota_1 = /\iota_2$ , the sign being so taken that  $\iota_1/\iota_1 = \iota_1\iota_2 = 1$  and  $\iota_2/\iota_2 = -\iota_2\iota_1 = 1$ .

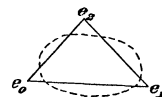


If  $a$  and  $b$  be any vectors we easily find that  $/a$  is  $\perp$  to  $a$ , and that  $a/b = TaTb \cos^2_a$ ; so that  $a/b = 0$  is the condition of perpendicularity between  $a$  and  $b$ .

Grassmann calls  $a/b$  the *inner* product of  $a$  and  $b$ , but it is also the *outer* product of  $a$  and  $/b$ , which is generally the most convenient way to regard it. We have also  $a/a = T^2a \cos 0 = T^2a = a^2$ , the last form being called the *inner square* of  $a$ , and written with the exponent underscored to distinguish from  $a^2 = aa$  which is always zero.

It will be seen that  $a/b$  is the same as Hamilton's  $Sab$  except that it is *positive* for values of the angle between  $a$  and  $b$  less than  $90^\circ$ , a manifest advantage.

Similarly all the points in a plane region can be expressed in terms of three given points  $e_0, e_1, e_2$ , and these may be so taken that  $e_0e_1e_2 = 1$ , then



$$\begin{aligned} /e_0 &= e_1e_2, \text{ so that } e_0/e_0 = e_0e_1e_2 = 1, \\ /e_1 &= e_2e_0, \quad \text{“} \quad e_1/e_1 = e_1e_2e_0 = e_0e_1e_2 = 1, \\ /e_2 &= e_0e_1, \quad \text{“} \quad e_2/e_2 = e_2e_0e_1 = e_0e_1e_2 = 1. \end{aligned}$$

If an ellipse be so drawn that  $e_1e_2$  is in it the anti-polar of  $e_0$ ,  $e_2e_0$  the anti-polar of  $e_1$ , and  $e_0e_1$  the anti-polar of  $e_2$ ; then, if  $p$  be any point whatever,  $/p$  is its anti-polar in this same ellipse.

In *space* we have similarly, if  $i_1, i_2, i_3$  form a *unit normal system*,

$$/i_1 = i_2i_3, \text{ so that } i_1/i_1 = i_1i_2i_3 = 1, \text{ etc., etc.,}$$

$/a$  is now a *plane-vector*  $\perp$  to  $a$ , and  $a/b$  has the same meaning as before.

In a *point* system  $e_0, e_1, e_2, e_3$  to which all points in space may be referred, and which may be so taken that  $e_0e_1e_2e_3 = 1$ , we have

$$\begin{aligned} /e_0 &= e_1e_2e_3, \text{ so that } e_0/e_0 = e_0e_1e_2e_3 = 1, \\ /e_1 &= -e_2e_3e_0, \quad \text{“} \quad e_1/e_1 = -e_1e_2e_3e_0 = e_0e_1e_2e_3 = 1, \text{ etc., etc.} \end{aligned}$$

Any point  $p$  is the anti-pole of its complement  $/p$  in an ellipsoid, so taken that in it  $/e_0$  is the anti-polar plane of  $e_0$ , etc.

One of the great advantages of this system over quaternions is, that while better adapted for treating geometry by the use of vectors only, it is also, when a *point* system is used, precisely fitted for bringing out polar-reciprocal relations. For if we have any equations expressing geometric relations in terms of points and lines in a plane region, or points, lines and planes in space, we have only in the first case to put lines for points and points for lines, and in the second to put

planes for points and points for planes, in order to obtain the polar reciprocal relations.

A few examples will now be presented to afford a comparison between this system and quaternions.

We will first give the equation by which Tait, in Art. 247 of his Quaternions, proves Pascal's Theorem together with the equation of precisely the same meaning in Grassmann's notation. They are respectively

$$S.V(V\alpha\beta V\delta\varepsilon)V(V\beta\gamma V\varepsilon\rho)V(V\gamma\delta V\rho\alpha)=0$$

and

$$(\alpha\beta.\delta\varepsilon)(\beta\gamma.\varepsilon\rho)(\gamma\delta.\rho\alpha)=0.$$

The comparative simplicity of the latter is apparent at a glance, and the ease of interpretation is as much greater as the labor of writing is less. There are ten capital letters in the first which are dispensed with in the last.

The equation is that of a cone on which lie each of the five vectors  $\alpha, \beta, \gamma, \delta, \varepsilon$  drawn outwards from a common point, and is also the condition that the three vectors  $\alpha\beta.\delta\varepsilon, \beta\gamma.\varepsilon\rho, \gamma\delta.\rho\alpha$  shall lie in one plane. Pascal's theorem is proved by considering a *section* of this cone.

If, however, we use *points* instead of vectors we have just what we want, the equation of a *conic* through five points, and the proof of the theorem follows immediately. If  $e_1, e_2, e_3, e_4, e_5$  are five points, the conic passing through them may be written

$$(pe_1.e_3e_4)(e_1e_2.e_4e_5)(e_2e_3.e_5p)=0,$$

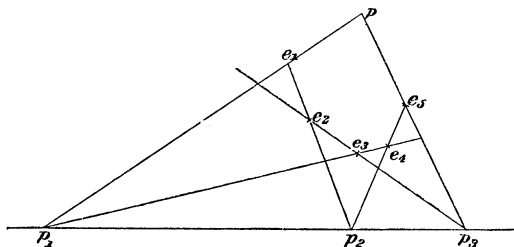
which is also the condition that the three points in parentheses shall lie on one right line.

This equation possesses the further advantage that by merely substituting  $L$ 's for  $p$ 's and  $e$ 's we have the equation

$$(LL_1.L_3L_4)(L_1L_2.L_4L_5)(L_2L_3.L_5L)=0$$

which is the line equation of a conic tangent to five given lines, and gives at once Brianchon's theorem.

There are given two points  $e_0$  and  $e_1$ , and two lines  $e_2e_4$  and  $e_3e_5$ ; two lines pass constantly through  $e_0$  and  $e_1$  respectively, and their points of intersection with  $e_2e_4$  and  $e_3e_5$  move along these lines with velocities bearing a constant ratio to each other: to find the locus of  $p$  the common point of the two moving lines.



## BY QUATERNIONS.

Let  $\overline{e_0 e_2} = a\varepsilon_1 + b\varepsilon_2 = \varepsilon_4$ ,  $\overline{e_2 e_4} = \varepsilon_2$ ,  $\overline{e_5 e_3} = \varepsilon_3$ ,  $\overline{e_0 e_1} = c\varepsilon_1$ ,  $\overline{e_1 e_3} = f\varepsilon_1$  and  $\varepsilon_3 = m\varepsilon_1 + n\varepsilon_2$ : then  $\rho = u(a\varepsilon_1 + b\varepsilon_2 + x\varepsilon_2) \dots (\alpha)$  and  $\rho = c\varepsilon_1 + v[f\varepsilon_1 + x(m\varepsilon_1 + n\varepsilon_2)] \dots (\beta)$

Operate on  $(\alpha)$  by  $V_{\varepsilon_1}$  and  $V_{\varepsilon_2}$  successively and divide thus

$$V_{\varepsilon_1} \rho = u(b+x) V_{\varepsilon_1} \varepsilon_2,$$

$$V_{\varepsilon_2} \rho = -au V_{\varepsilon_1} \varepsilon_2,$$

$$\text{and } \frac{V_{\varepsilon_1} \rho}{V_{\varepsilon_2} \rho} = \frac{b+x}{-a} : \therefore x = \frac{V(a\varepsilon_1 + b\varepsilon_2) \rho}{-V_{\varepsilon_2} \rho} \\ = -\frac{V_{\varepsilon_4} \rho}{V_{\varepsilon_2} \rho}.$$

Similarly from  $(\beta)$

$$V_{\varepsilon_1} \rho = vx n V_{\varepsilon_1} \varepsilon_2,$$

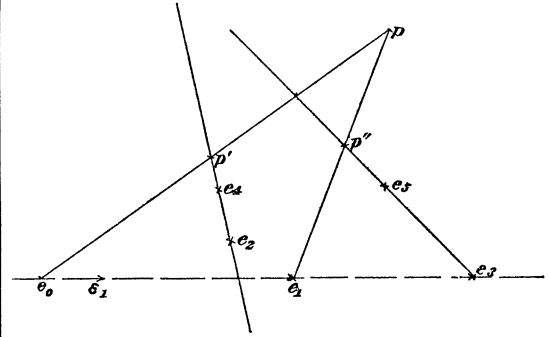
$$V_{\varepsilon_2} \rho = -c V_{\varepsilon_1} \varepsilon_2 - v(f+mx) V_{\varepsilon_1} \varepsilon_2,$$

$$\therefore \frac{V_{\varepsilon_1} \rho}{V_{\varepsilon_2} (\rho - c\varepsilon_1)} = \frac{-nx}{f+mx} = \frac{n \frac{V_{\varepsilon_4} \rho}{V_{\varepsilon_2} \rho}}{f-m \frac{V_{\varepsilon_4} \rho}{V_{\varepsilon_2} \rho}}$$

$$\text{or } \frac{V_{\varepsilon_1} \rho}{V_{\varepsilon_2} (\rho - c\varepsilon_1)} = \frac{n V_{\varepsilon_4} \rho}{V_{(\varepsilon_2 f - m\varepsilon_4) \rho}}$$

the equation of a conic because of the second degree in  $\rho$ .

## BY CALCULUS OF POSITION.



$p' = e_2 + x(e_4 - e_2)$ ,  $p'' = e_3 + x(e_5 - e_3)$  and also  $e_0 p' p = 0$ ,  $e_1 p'' p = 0$ .

Insert values of  $p'$  and  $p''$ ,

$$\therefore e_0 e_2 p + x e_0 (e_4 - e_2) p = 0, \text{ or } x = \frac{-e_0 e_2 p}{e_0 (e_4 - e_2) p}$$

and

$$e_1 e_3 p + x e_1 (e_5 - e_3) p = 0, \text{ or } x = \frac{-e_1 e_3 p}{e_1 (e_5 - e_3) p}$$

$\therefore$  equating values of  $x$

$$\frac{e_0 e_2 p}{e_0 (e_4 - e_2) p} = \frac{e_1 e_3 p}{e_1 (e_5 - e_3) p}$$

a conic because of the second degree in  $p$ .

From the complementary equation,

$$\frac{L_0 L_2 L}{L_0 (L_4 - L_2) L} = \frac{L_1 L_3 L}{L_1 (L_5 - L_3) L}$$

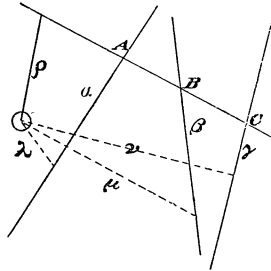
we may derive at once a polar reciprocal theorem.

Working with vectors but using Grassmann's system we should have saved the writing of twenty-seven unnecessary  $V$ 's.

As one more example we will take the following. To find the equation of the surface generated by a right line moving on three right lines as directrices.

BY QUATERNIONS.

$ABC$  is a generatrix of the surface, its equation being either



$$\rho = \lambda + x\alpha + u(\mu + y\beta - \lambda - x\alpha) \dots (1)$$

or

$$\rho = \mu + y\beta + v(v + z\gamma - \mu - y\beta), \dots (2)$$

from (1) we have

$$S\alpha\beta\rho = S\alpha\beta\lambda + u(S\alpha\beta\mu - S\alpha\beta\lambda);$$

$$\therefore u = \frac{S\alpha\beta(\rho - \lambda)}{S\alpha\beta(\mu - \lambda)}, \dots (3)$$

Similarly from (2)

$$v = \frac{S\beta\gamma(\rho - \mu)}{S\beta\gamma(\nu - \mu)} \dots (4)$$

Operate on (1) by  $S.V\gamma\alpha$

$$\therefore S\gamma\alpha\rho = S\gamma\alpha\lambda + uS\gamma\alpha(\mu + y\beta - \lambda)$$

$$\therefore y = \frac{S\gamma\alpha(\rho - \lambda) - uS\gamma\alpha(\mu - \lambda)}{uS\alpha\beta\gamma} \text{ [and by (3)]}$$

$$y = \frac{S\gamma\alpha(\rho - \lambda)S\alpha\beta(\mu - \lambda) - S\gamma\alpha(\mu - \lambda)S\alpha\beta(\rho - \lambda)}{S\alpha\beta\gamma S\alpha\beta(\rho - \lambda)}$$

$$= \frac{S.V\gamma\alpha V\alpha\beta V(\mu - \lambda)(\rho - \lambda)}{S\alpha\beta\gamma S\alpha\beta(\rho - \lambda)}.$$

Operating on (2) by  $S.V\gamma\alpha$  and substituting value of  $v$  we have by a similar reduction

$$y = \frac{S.V\beta\gamma V\gamma\alpha V(\nu - \mu)(\rho - \mu)}{S\alpha\beta\gamma S\beta\gamma(\rho - \nu)}.$$

Equating these values of  $y$  we have the equation of the surface of the second degree in  $\rho$ , viz.:

$$\frac{S.V\gamma\alpha V\alpha\beta V(\mu - \lambda)(\rho - \lambda)}{S\alpha\beta(\rho - \lambda)}$$

$$= \frac{S.V\beta\gamma V\gamma\alpha V(\nu - \mu)(\rho - \mu)}{S\beta\gamma(\rho - \nu)}$$

BY CALCULUS OF POSITION.

Let the three lines be  $L_1, L_2, L_3$ . If through a point on  $L_2$  planes be passed containing  $L_1$  and  $L_3$ , then their common line will be a generatrix of the surface, for it will evidently cut  $L_1, L_2$  and  $L_3$ . Let  $p$  be any point of this common line, then the two planes will be  $pL_1$  and  $L_3p$ . These are to cut  $L_2$  at the same point, the condition for which is

$$pL_1.L_2.L_3p = 0.$$

This being a scalar equation of the second degree in  $p$  represents a quadric, which is the required surface.

In working out the vector equation with the other system we should save the writing of *forty-six* unnecessary  $S$ 's and  $V$ 's. The last equation, for instance, would be

$$\frac{\gamma\alpha.\alpha\beta.(\mu-\lambda)(\rho-\lambda)}{\alpha\beta(\rho-\lambda)} = \frac{\beta\gamma.\gamma\alpha.(\nu-\mu)(\rho-\mu)}{\beta\gamma(\rho-\nu)}.$$

To illustrate the applicability of this method to Mechanics we will give a single example.

Let  $P_1, P_2$ , etc., be vectors representing in magnitude and direction any system of forces acting on a rigid body, and let  $e_1, e_2$ , etc., be points in their respective lines of action, then the forces are completely represented by  $e_1 P_1, e_2 P_2$ , etc.

The resultant action of the system is simply the sum of these forces, viz.  $\sum_1^n (e P)$ : and for equilibrium the condition is

$$\sum_1^n (e P) = 0.$$

Add and subtract  $e_0 \Sigma(P)$ ,  $e_0$  being any point whatever ;

$$\sum_1^n (e P) - e_0 \sum_1^n (P) + e_0 \sum_1^n (P) = 0$$

or

$$e_0 \sum_1^n (P) + \sum_1^n [(e - e_0) P] = 0.$$

But one of these terms is a *point-vector*, and the other a *plane-vector*; therefore, being quantities of different kinds, in order to satisfy the equation they must be *separately* equal to zero.

$$\therefore \left. \begin{array}{l} \Sigma(P) = 0 \\ \Sigma[(e - e_0) P] = 0 \end{array} \right\}$$

The first signifies that the resultant *force* must be zero, the second that the resultant *couple* must be zero.

In closing I may remark that a principal reason for the slow introduction of Grassmann's method seems to be the great *generality* of his demonstrations as given in his books. They being usually given for space of the  $n^{\text{th}}$  order, the idea appears to have prevailed that the method is peculiarly adapted to *hyper-geometry*, which is actually the case, without, however, at all interfering with its special fitness for application to space of two or three dimensions.